On the estimates to the eigenvalues of a Robin problem

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We are interested in the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \quad \text{on} \quad \Gamma, \tag{2}$$

in the bounded domain $\Omega \subset R^n$, $n \geq 2$. Here we assume the boundary surface Γ belong to C^2, ν is the outward unit normal vector to Γ and α is a real parameter. The problem (1), (2) is usually referred to as Robin problem for $\alpha > 0$ and as generalized Robin problem for $\alpha < 0$.

We have the sequence of eigenvalues $\lambda_1(\alpha) < \lambda_2(\alpha) \leq \cdots \to +\infty$ enumerated according to their multiplicities where $\lambda_1(\alpha)$ is simple with a positive eigenfunction. Let $0 < \lambda_1^D < \lambda_2^D \leq \cdots \to +\infty$ are the eigenvalues of the Dirichlet eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \Gamma.$$

Theorem 1. *The eigenvalues $\lambda_k(\alpha)$, $k = 1, 2, \ldots$ have the following properties:*
i) $\lambda_k(\alpha_1) < \lambda_k(\alpha_2) < \lambda_k^D$ *for* $\alpha_1 < \alpha_2$*;*
ii) $\lambda_k(\alpha)$ *is a continuous function of* α*;*
iii) $\lambda_k(\alpha)$ *is a concave function of* α*:*

$$\lambda_k(\beta\alpha_1 + (1-\beta)\alpha_2) \geq \beta\lambda_k(\alpha_1) + (1-\beta)\lambda_k(\alpha_2), \quad 0 < \beta < 1.$$

Theorem 2. *The eigenvalues $\lambda_k(\alpha)$, $k = 1, 2, \ldots$ obey the estimates*

$$0 < \lambda_k^D - \lambda_k(\alpha) \leq C\alpha^{-1/2} \left(\lambda_k^D\right)^2, \quad \alpha > 0,$$

where the constant C depends on the domain Ω and does not depend on k.

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