Semidefinite optimization for measure-valued differential equations

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The Liouville equation, also called continuity equation, or advection equation, or equation of conservation of mass, is a classical linear partial differential equation arising in fluid mechanics, statistical physics or kinetic theory. We are interested in numerical methods for constructing weak, or measure-valued solutions to the Liouville equation in integral form:

$$\frac{\partial \mu}{\partial t} + \mathrm{div}\, f\mu = \mu_0 - \mu_1$$

where measure μ disintegrates into $\mu(\mathrm{d}t, \mathrm{d}x, \mathrm{d}u) = \lambda(\mathrm{d}t)\, \xi(\mathrm{d}x \,|\, t)\, \omega(\mathrm{d}u \,|\, t, x)$ where $\lambda(\mathrm{d}t)$ is the uniform probability measure on the time interval $[0, 1]$, $\xi(\mathrm{d}x \,|\, t)$ is a time-dependent probability measure (also called Young measure) ruling the distribution of a state x in a set $X \subset \mathbb{R}^n$, and $\omega(\mathrm{d}u \,|\, t, x)$ is a time- and state-dependent probability measure (also called relaxed control) ruling the distribution of a control u in a set $U \subset \mathbb{R}^m$. Probability measures $\mu_0(\mathrm{d}t, \mathrm{d}x) = \delta_0(\mathrm{d}t)\, \xi(\mathrm{d}x \,|\, t = 0)$ and $\mu_1(\mathrm{d}t, \mathrm{d}x) = \delta_1(\mathrm{d}t)\, \xi(\mathrm{d}x \,|\, t = 1)$ correspond to boundary conditions at initial and terminal time, and they can be given, or unknown, depending on the problem. The Liouville equation models the transportation of a state probability distribution on X from $\xi_0(\mathrm{d}x) = \xi(\mathrm{d}x \,|\, t = 0)$ to $\xi_1(\mathrm{d}x) = \xi(\mathrm{d}x \,|\, t = 1)$ along the flow of the (convexified and relaxed) controlled ordinary differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t) \in X, \quad u(t) \in U, \quad t \in [0, 1].$$

In this contribution we survey the application of convex optimization techniques to solve decision problems involving the Liouville equation. Under the assumption that the vector field f is polynomial and the state and control constraint sets X and U are compact basic semialgebraic, we show that semidefinite programming, or linear programming in the convex cone of nonnegative quadratic forms, can be used to provide constructive numerical solutions to the problem of optimizing linear functionals of measure-valued solutions to the Liouville equation. Applications range from the estimation of the region or basin of attraction of a given target set of a controlled ordinary differential equation, to nonconvex polynomial optimal control under nonconvex state constraints, including e.g. the dynamic formulation of the Monge-Kantorovich optimal mass transportation problems with semialgebraic costs and obstacles.

References

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