Abstract size-structured population dynamics in Banach spaces

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Of concern is a system of abstract partial differential equations in Banach spaces associated with a system of size-structured population models with spacial diffusions. In particular, application to an epidemic model with spacial diffusion is presented.

Let X be a Banach space and denote by $L^1(X^N)$ the space of Bochner integrable functions from $(0, s_\dagger)$ to the product Banach space X^N. For each $i = 1, \ldots, N$, let A^i be the infinitesimal generator of a (C_0)-semigroup $\{T^i(t) : t \geq 0\}$ in X. We consider $g^i : [0, \infty) \times X^N \to [0, \infty)$ as the growth rate function and $\mu^i : [0, s_\dagger) \to [0, \infty)$ as the mortality rate function. To describe the birth process, we employ functions $\beta^{ij} : (0, s_\dagger) \times X^N \to [0, \infty)$ for $i, j = 1, 2 \ldots, N$.

For given initial value $p_0 = (p_0^1, \ldots, p_0^N) \in L^1(X^N)$, we consider the following initial boundary value problem of abstract partial differential equations in X:

$$\text{(AP)} \qquad \begin{cases} \partial_t p^i(s,t) + \partial_s(g^i(s, P(t))p^i(s,t)) - A^i p^i(s,t) \\ \qquad\qquad = -\mu^i(s)p^i(s,t) + G^i(p(\cdot,t))(s), \quad s \in [0, s_\dagger),\ t \in [0,T], \\ g^i(0,t)p^i(0,t) = \displaystyle\sum_{j=1}^{N} \int_0^{s_\dagger} \beta^{ij}(s, P(t))p^j(s,t)\,\mathrm{d}s, \quad t \in [0,T], \\ \qquad P(t) = (P^1(t), \ldots, P^N(t)), \quad P^i(t) = \displaystyle\int_0^{s_\dagger} p^i(s,t)\,\mathrm{d}s \\ \quad p^i(s,0) = p_0^i(s), \quad s \in [0, s_\dagger). \end{cases}$$

We show the existence of a unique mild solution $p = (p^1, \ldots, p^N) \in C([0,T]; L^1(X^N))$ to (AP).

As an application, we consider the SIR epidemic model with spacial diffusion. We take $N = 3$ and let $p^1(s,t,x)$, $p^2(s,t,x)$ and $p^3(s,t,x)$ be the population density with respect to $s \in [0, s_\dagger)$ and position $x \in \Omega$ at time $t \in [0,T]$ in the susceptible class (S), the infective class (I), and the removed class (R), respectively. Here Ω is the habitat of the biological population and we set $X = L^1(\Omega)$ and define the operator A^i by the Laplacian with suitable boundary condition. We assume that the infective rate and the recovery rate depend on the total population $P^2(t)$ in the infective class (I) as well as size s. Denoting by $\Phi(s, P^2(t))$ and $\Psi(s, P^2(t))$ the infective rate and the recovery rate, respectively, we put $G^1(p(\cdot,t))(s) = -\Phi(s, P^2(t))p^1(s,t,x)$, $G^2(p(\cdot,t))(s) = -\Psi(s, P^2)p^2(s,t,x) + \Phi(s, P^2(t))p^1(s,t,x)$, $G^3(p(\cdot,t))(s) = \Psi(s, P^2)p^2(s,t,x)$. As the birth law, we assume that members in every classes can reproduce zero-size individuals, but the zero size individuals belong to susceptible class (S) and there is no zero-size individual in infective class (I) nor in removed class (R). Thus we take $\beta^{1j}(s, P(t)) = \beta^{1j}(s, P^j(t))$ and $\beta^{2j}(s, P(t)) = \beta^{3j}(s, P(t)) = 0$ for $j = 1, 2, 3$.