Forward-backward diffusion equations and indefinite spectral problems

Aleksey Kostenko
University of Vienna, Austria
Oleksiy.Kostenko@univie.ac.at

We consider the following parabolic forward-backward equation

$$(1)\qquad (\operatorname{sgn} x)w(x)u_t(x,t) = (r(x)u_x(x,t))_x - q(x)u(x,t), \quad (0 < t < +\infty,\ x \in (-a,a)).$$

Here $q, r^{-1}, w \in L^1_{\mathrm{loc}}(-a,a)$ satisfy $w, r > 0$ and $q \geq 0$ a.e. on $(-a,a) \subseteq \mathbb{R}$. It is assumed that the function u satisfies

$$(2)\qquad u(x,0) = \phi_+(x), \quad x \in (0,a); \quad \int_{-a}^{a} |u(x,t)|^2 w(x)\mathrm{d}x = O(1), \quad t \to \infty.$$

It is also assumed that $\phi_+ \in L^2_w(0,a)$. Moreover, if necessary additional self-adjoint t-independent boundary conditions at $x = -a$ and $x = a$ are assumed.

Boundary value problems of this (forward-backward) type arise in kinetic theory and in the theory of stochastic processes. Separation of variables in (1) leads to the indefinite spectral problem

$$(3)\qquad -(r(x)y')' + q(x)y = \lambda\, (\operatorname{sgn} x)w(x)y, \quad x \in (-a,a),$$

and the well-posedness issue for the boundary value problem (1)–(2) is closely connected with the similarity problem for the corresponding indefinite Sturm-Liouville operator $A = \frac{(\operatorname{sgn} x)}{w}(-\frac{\mathrm{d}}{\mathrm{d}x} r \frac{\mathrm{d}}{\mathrm{d}x} + q)$ acting in $L^2_w(-a,a)$. Assuming that the functions q, r, w are even, we give a criterion formulated in terms of coefficients q, r, w for the similarity of A to a self-adjoint operator. Then we discuss the solvability of (1)–(2) by using separation of variables. A connection with the Hardy-Littlewood-Polya-Everitt (HELP) inequality will be mentioned too.

References

[1] *A. Kostenko*: On a necessary aspect for the Riesz basis property for indefinite Sturm-Liouville problems. ArXiv:1202.2444, 26 pp.

[2] *A. Kostenko*: The similarity problem for indefinite Sturm-Liouville operators and the HELP inequality. ArXiv:1207.2586, 42 pp.