Smooth attractors for quintic wave equations with fractional damping

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We study wave equations in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ with critical quintic nonlinearity and damping term involving the fractional Laplacian:

$$\partial_t^2 u + \gamma(-\Delta)^{\frac{1}{2}}\partial_t u - \Delta u + f(u) = g(x), \quad x \in \Omega \tag{1}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad u|_{\partial\Omega} = 0, \tag{2}$$

where $\gamma > 0$, $g(x) \in L^2(\Omega)$, $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $f \in C^1(\mathbb{R})$ satisfies the following growth assumptions

$$-C_1 \le f'(s) \le C_2(1 + |s|^4), \tag{3}$$

with some positive constants C_1, C_2.

In this work we establish the additional regularity of *any* energy solution of problem (1)–(2)

$$u \in L^2(0, T; H^{\frac{3}{2}}(\Omega)), \tag{4}$$

where $H^s(\Omega) = D((-\Delta)^{\frac{s}{2}})$. Notice that this regularity does not follow from energy estimates. As not difficult to see the desired regularity would follow if we succeed to multiply equation (1) by $(-\Delta)^{\frac{1}{2}}u$ and estimate non-linear term with fractional Laplacian: $(f(u), (-\Delta)^{\frac{1}{2}}u)$. It appears that similarly to estimates $(f(u), u) \ge -C\|u\|^2$ and $(f(u), u) \ge -C\|u\|^2_{H^1}$ we are able to prove the following inequality

$$(f(u), (-\Delta)^{\frac{1}{2}}u) \ge -C\|u\|^2_{H^{\frac{1}{2}}(\Omega)}, \tag{5}$$

that holds at least when Ω is torus or the whole space $\mathbb{R}^3$. The case of arbitrary bounded smooth domain requires some additional technical work involving extension arguments and estimates of corrector terms, nevertheless we are still able to prove (4).

Extra regularity (4) allows us to verify the global well-posedness of problem (1)–(2) with critical quintic non-linearity which was a long standing problem (see [1], [2]). Also based on (4) we are able to prove that dynamical system generated by (1)–(2) possesses a smooth compact global attractor of finite Hausdorff and fractal dimension.

References

[1] *A. Carvalho, J. Cholewa*: Attractors for strongly damped wave equations with critical nonlinearities. Pac. J. Math. *207* (2002), 287–310.

[2] *V. Kalantarov, S. Zelik*: Finite-dimensional attractors for the quasi-linear strongly-damped wave equation. J. Differ. Equation *247* (2009), 1120–1155.