# Uniqueness of positive radial solutions of $\Delta u + g(r)u + h(r)u^p = 0$ and its applications


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In this talk, we consider the problem

$$
\begin{cases} u_{rr} + \dfrac{n-1}{r}u_r + g(r)u + h(r)u^p = 0, \quad 0 < r < R, \\ u(0) \in (0,\infty), \quad u_r(0) = 0, \quad u(R) = 0, \end{cases} \tag{1}
$$

where $p > 1$, $n \geq 2$, $R \in (0,\infty]$, $g \in C^1((0,\infty))$ and $h \in C^3((0,\infty))$. If $R = \infty$, $u(R) = 0$ means $u(r) \to 0$ as $r \to \infty$. By introducing a new generalized Pohožaev identity, we give a uniqueness result of the positive solutions of (1). We also study annular domain or exterior domain cases.

As applications, we consider the uniqueness of positive solutions of the following problems with $n \geq 3$ and $1 < p \leq (n+2)/(n-2)$.

(i) The Brezis-Nirenberg problem

$$
-\Delta_{S^n} u - \lambda u = u^p \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{2}
$$

where $D(\subset S^n)$ is a geodesic ball whose center is the north pole, $\Delta_{S^n}$ is the Laplace-Beltrami operator on $S^n$, $\lambda < \lambda_1$, and $\lambda_1$ is the first eigenvalue of $\Delta_{S^n}$ on $D$.

(ii) A nonlinear Schrödinger equation with harmonic potential

$$
\Delta u - (\lambda + |x|^2)u + u^p = 0 \quad \text{in } \mathbb{R}^n, \qquad u \in \Sigma, \tag{3}
$$

where $\lambda > -n$ and $\Sigma = \{v \in H^1(\mathbb{R}^n) : |x|v \in L^2(\mathbb{R}^n)\}$.

(iii) The Haraux-Weissler equation

$$
\Delta u(x) + \frac{1}{2}x \cdot \nabla u(x) + \lambda u(x) + u(x)^p = 0 \quad \text{in } \mathbb{R}^n, \qquad u \in H^1_\rho(\mathbb{R}^n), \tag{4}
$$

where $H^1_\rho(\mathbb{R}^n) = \left\{u \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2\right) \rho(x)\, dx < \infty\right\}$ with $\rho(x) = \exp(|x|^2/4)$.
By using our generalized Pohožaev identity, we also show the nonexistence of a positive solution of (3) in the case of $n = 3$ and $p = 5$.


This is a joint work with Professor Kohtaro Watanabe.