An ergodic Poincaré-Bendixson theorem for extended scalar reaction-diffusion equations

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We consider the autonomous scalar reaction-diffusion equation

$$u_t = u_{xx} + f(x, u, u_x), \tag{1}$$

where f is C^2, 1-periodic in x, and more generally a non-autonomous equation

$$u_t = u_{xx} + f(t, x, u, u_x), \tag{2}$$

where f is also 1-periodic in t. We consider it as an extended system, analyzing time evolution of functions $u : \mathbf{R} \to \mathbf{R}$ not necessarily either spatially periodic or decaying at infinity. More precisely, the phase space is $X = H^2_{ul}(\mathbf{R})$, which stands for a uniformly local space with the appropriate weighting function ρ. We consider the state space Y of all $u \in X$ such that $u(t)$ is uniformly bounded in X, equipped with the local (i.e. weighted) $C^1_\rho(\mathbf{R})$ topology.

Our ergodic version of the Fiedler and Mallet-Paret Poincaré-Bendixson theorem for extended systems (1), (2) now reads ([3]): The union of supports $\mathcal{A}$ of all the space-time invariant measures of (1), (2) on Y projects injectively to a dynamical system on $\mathbf{R}^2$. In the case of (1), $\mathcal{A}$ consists of only equilibria and periodic orbits. A consequence is that the space-time topological entropy of (1) is zero, which complements well recent results of Mielke, Turaev and Zelik.

As a corollary, we partially recover a version of Poincaré-Bendixson theorem for (1), (2) on a bounded domain with periodic boundary conditions, and show that the union of $\overline{\omega}(u)$ projects injectively to $\mathbf{R}^2$. Here $\overline{\omega}(u)$ is the essential ω-limit set, namely a subset of $\omega(u)$ such that for each open neighborhood U of $\overline{\omega}(u)$, the Banach density of times such that $u(t) \in U$ is 1.

An application is that we obtain a canonical construction of a vector field on $\mathbf{R}^2$ associated to (1), and so give a rigorous foundation for the Joly-Raugel correspondence [1]. Finally, we comment on the discrete-space analogue of the results [2].

References

[1] *R. Joly, G. Raugel*: A striking correspondence between the dynamics generated by the vector fields and by the scalar parabolic equation. Confluentes Math. *3* (2011), 471–493.

[2] *S. Slijepčević*: The Aubry-Mather theorem for driven generalized elastic chains. Preprint, arXiv 1305.1109.

[3] *S. Slijepčević*: An ergodic Poincaré-Bendixson theorem for extended scalar reaction-diffusion equations. In preparation.