Existence of solutions of integral equations related to inverse problems of quasilinear ordinary differential equations

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Let $m > 1$, $f \in C((0,\infty);(0,\infty))$ and consider the following IVP:

$$(|u'|^{m-1}u')' + f(u) = 0, \tag{1}$$

$$u'(0) = 0, \quad \text{and} \quad u(0) = p, \tag{2}$$

where $p > 0$ is a parameter. Then we can show that there is a unique time $T(p) \in (0,\infty)$ satisfying

$$u(t) > 0 \text{ on } [0, T(p)) \quad u(T(p)) = 0; \quad \text{and}\, u'(t) < 0 \text{ in } (0, T(p)). \tag{3}$$

In fact, $T(p)$ is explicitly given by means of p, m and $f(u)$ in the form:

$$T(p) = \Big(\frac{m}{m+1}\Big)^{1/(m+1)} \int_0^p \bigg(\int_v^p f(z)\,\mathrm{d}z\bigg)^{-1/(m+1)} \mathrm{d}v. \tag{4}$$

For example, if $m = 1$ and $f(u) = u^\alpha$, $\alpha > 0$, then $T(p)$ is given by

$$T(p) = \frac{1}{\sqrt{2(\alpha+1)}}\, B\Big(\frac{1}{\alpha+1}, \frac{1}{2}\Big)\, p^{(1-\alpha)/2}.$$

We want to consider the following inverse problem: for given function $T(p)$, are there nonlinearities $f(u)$ of IVP (1)–(2) which surely realize the relation (3)?

By the preceding observation we find that, under suitable conditions, this problem is equivalent to solving the integral equation (4) whose unknown is $f(u)$.

Theorem 1. *Let $T(r)$ be a positive and Lipschitz continuous function defined on $[0,R]$. Then there exists a unique solution f of (4) that is continuous on $[0,R]$, and positive on $(0,R]$.*

Theorem 2. *Let $T(r)$ be a positive and Hölder continuous function defined on $[0,R]$ with exponent λ, $m/(m+1) \le \lambda < 1$. Then there exists a solution f of (4) that is continuous on $[0,R]$, and positive on $(0,R]$.*

References

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