Multigrid for Helmholtz equations just does not work! Or does it?

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The development of multigrid methods in the 1960s represents a milestone for solving linear systems of equations obtained by discretizing PDEs, allowing such a system to be solved with the optimal work and storage complexity, i.e., proportional to the number of unknowns. While enormously effective when applied to coercive equations as well as a multitude of more difficult problems, extending a standard multigrid method for solving the Helmholtz equation with the same efficiency remains an open problem. In fact, all iterative methods have severe convergence problems when applied to the discretized Helmholtz equation, for an overview, see [1].

A particular approach that received a lot of attention over the last decade is the shifted Laplace preconditioner [2], see also [3]. The idea is to shift the wave number into the complex plane, just enough to make multigrid work on the shifted problem, and then to use the shifted problem as a preconditioner to solve the original Helmholtz equation using a Krylov method. There are however two conflicting requirements: the shift should be large enough in order for multigrid to work, and not too large, in order to still have a good preconditioner of the original problem.

We first show a rigorous result on the size of the shift, for which we can prove that the shifted problem is still a good preconditioner [4]. This shift is however not enough for multigrid to work [1]. We then explain in detail what happens in the standard multigrid algorithm when one applies it to the Helmholtz equation [5]. We present a rigorous convergence analysis of a two grid cycle, and show that it is possible in one spatial dimension to obtain a convergent algorithm using just standard components. We illustrate our results with numerical experiments, and also discuss the difficulties to extend these results to higher spatial dimensions.

References

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