On adaptivity, convergence and inexact algebraic computations in numerical solution of partial differential equations

Zdeněk Strakoš
Faculty of Mathematics and Physics, Charles University in Prague, Czech Republic
strakos@karlin.mff.cuni.cz

Problems in sciences and engineering described by partial differential equations are often solved numerically using discrete approximations of the infinite dimensional formulations via the Finite Element Method (FEM). Such approximations presume *locally supported* basis functions which results in algebraic problems with *sparse matrices.* Locality of the FEM bases and sparsity of the resulting matrices is considered an advantage over some alternative approaches. In many problems, however, one can observe a fast *global exchange* of information over the domain. This phenomenon is not reflected by the locality of the FEM bases and by the sparsity of the resulting matrices representing the discretized operators; the global communication is to be achieved through numerical solving the (large and sparse) algebraic problems. Computational efficiency can then be obtained by incorporating coarse space components (e.g. using domain decomposition or multilevel methods). Efficient preconditioning can be seen as another tool for achieving the same goal. Preconditioning should (ideally) reflect the physical nature of the problem expressed in the mathematical model. It can be conveniently motivated using a functional analytic operator description (operator preconditioning).

In solving challenging problems, *adaptivity* is the key concept. It takes the form of an adaptive mesh refinement (and, if applicable, also time step refinement) as well as of adaptation of algebraic solvers to data in the process of computation. The investigation of *the rate of convergence* is often based on seeing individual steps (adaptation cycles, mesh refinement or individual algebraic iterations) as contractions for some error estimators, where the contraction parameter is independent of the individual step. This views convergence *asymptotically* and it seemingly allows reaching an arbitrary prescribed accuracy in a finite number of contraction steps. In practical computations, however, asymptotic convergence bounds may not work well, and an arbitrary accuracy can not be reached simply due to the fact that the discretized algebraic problems cannot be solved exactly. The restriction of a properly measured maximal attainable accuracy can be for difficult problems significant.

The proper measure should concern errors in approximation of the physically meaningful quantities by the outputs of numerical computations. The use of residuals measuring the accuracy to which the computed quantities satisfy the governing equations is conditioned by sensitivity of the problem to be solved. In order to stop the solution process, we need a fully computable *a posteriori error estimators* which allow for the local error control based on the comparison of the size of the error from different sources (discretization, linearization, inexact algebraic computation).

This contribution will focus on several questions outlined above with emphasis on coupling algebraic computations and their analysis into the whole solution process. It greatly benefits from a joint work with Jörg Liesen, Martin Vohralík, Pavel Jiránek, Jan Papež and Tomás Gergelits.