The electrostatics problem with a dipole source: theoretical results and numerical approximation

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Aim of this talk is to give a simple and direct proof, by using the classical *duality method*, of the existence, uniqueness and numerical approximation of the solution to an electrostatics-like problem when the source (namely, the applied current density) is a current dipole.

This is related to the problem of electroencephalography: a non-invasive technique for detecting the brain activity from the measure of the electric field (or of its potential) on the head surface. In mathematical terms, an inverse problem in which one wants to determine the source, that has generated the electric field, by measuring the boundary value of the electric potential.

The first step of an inverse problem is to solve in an efficient way the forward problem: assuming, as it is usually done in this context, that the time-variation of the electric and magnetic fields is not relevant, the Maxwell equations lead to the following electrostatics problem with a dipole source:

$$\begin{cases} \operatorname{div}(\sigma \operatorname{grad} u) = \operatorname{div}(\mathbf{p}_0 \delta_{\mathbf{x}_0}) & \text{in } \Omega \\ (\sigma \operatorname{grad} u) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here Ω is a bounded domain in $\mathbf{R^3}$ (representing the human head), $\mathbf{n}$ the unit outward normal vector on $\partial\Omega$, u the (unknown) potential of the electric field, σ the electric conductivity (a symmetric and positive definite matrix, with non-constant entries), $\mathbf{p}_0$ the constant polarization, $\delta_{\mathbf{x}_0}$ the Dirac distribution centered at $\mathbf{x}_0$.

A suitable variational formulation of the singular problem (1) is devised, and a finite element approximation scheme is proposed and analyzed. In addition, a residual-type a-posteriori error estimator is determined, leading to an efficient and reliable adaptive procedure.

References

[1] *A. Valli*: Solving an electrostatics-like problem with a current dipole source by means of the duality method. Appl. Math. Lett. *25* (2012), 1410–1414.

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