Exposed solutions of differential inclusions

Vladimir V. Goncharov
CIMA-UE, Universidade de Évora, Portugal
goncha@uevora.pt

We present a new method for proving the existence of (Carathéodory) solutions to differential inclusions based on the properties of uniformly distributed *Brownian motion* and dual in some sense to the famous *Baire category approach.* Let us consider an (autonomous) differential inclusion

$$\dot{x} \in F(x) \tag{1}$$

where $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a bounded multifunction with compact and convex values. Due to applications in Optimal Control Theory it is important to approximate an arbitrary solution of (1) by a sequence of other solutions, whose derivatives admit as few as possible values from the right-hand side. In particular, the well known relaxation result is the so named (nonlinear) *bang-bang principle*, which asserts that the family $\mathcal{H}_{\mathrm{ext}\, F}(x_0)$ of solutions $x(\cdot)$, $x(0) = x_0$, to the extremal differential inclusion $\dot{x} \in \mathrm{ext}\, F(x)$ is dense (with respect to the uniform topology) in the respective solution set $\mathcal{H}_F(x_0)$ for the original convexified inclusion (1) whenever the multivalued mapping F is Lipschitz continuous. Moreover, in this case $\mathcal{H}_{\mathrm{ext}\, F}(x_0)$ can be represented as the intersection of a countable family of open dense sets in $\mathcal{H}_F(x_0)$. In other words, $\mathcal{H}_{\mathrm{ext}\, F}(x_0)$ is a "large" subset of $\mathcal{H}_F(x_0)$ in the topological sense, i.e., its compliment is a *meager set*. Recently, A. Bressan proposed in [1] a dual version of the Baire category method for proving just the existence of an extreme solution, where F is assumed to be merely continuous. Namely, he associates to each continuous function $p(\cdot)$ the set of solutions to (1) whose derivatives are orthogonal to $p(t)$ for a.e. t and applies the Baire cathegory argument on the space Ω of these continuous functions in the place of the solution set $\mathcal{H}_F(x_0)$. Supplying Ω with the Wienner measure and using some fine properties of Brownian paths we prove instead the existence of an *exposed solution* (remind that the set of exposed points is strictly included into the set of extreme ones already in $\mathbb{R}^2$) under the hypothesis that F is Hölder continuous with an exponent $\alpha > 1/2$. Furthermore, in the case when $F(x)$ is a nondegenerate interval moving in the lipschitzean way, say $F(x) = [f(x), g(x)]$, we construct a canonical probability measure supported on the set $\mathcal{H}_F(x_0)$ and such that *almost surely* (in the sense of this measure) the derivative $\dot{x}(t)$ is equal either to $f(x(t))$ or to $g(x(t))$ for a.e. t. The reaearch is originated by an idea of A. Bressan and fulfilled jointly with G. Colombo in a particular case $N = 2$ (see [2]).

References

[1] *A. Bressan*: Extremal solutions of differential inclusions via Baire theorem: a dual approach. Preprint, 2013.

[2] *G. Colombo, V. Goncharov*: Brownian motion and exposed solutions of differential inclusions. NoDEA, DOI 10.1007/s00030-012-0168-z (2012), to appear.