Optimal Hardy-type inequalities and the spectrum of the corresponding operator

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We give a general answer to the following fundamental problem posed by Shmuel Agmon 30 years ago:

> Given a (symmetric) linear elliptic operator P of second-order in R^n, find a continuous, nonnegative weight function W which is "as large as possible" such that for some neighborhood of infinity Ω_R the following inequality holds
>
> $$(P\phi, \phi) \geq \int_{\Omega_R} W(x)|\phi|^2 \, \mathrm{d}x \qquad \forall \phi \in C_0^\infty(\Omega_R).$$

We construct, for a *general* subcritical second-order elliptic operator P in a domain $\Omega \subset R^n$ (or a noncompact manifold), a Hardy-weight W which is *optimal* in the following sense. The operator $P - \lambda W$ is subcritical in Ω for all $\lambda < 1$, null-critical in Ω for $\lambda = 1$, and supercritical near any neighborhood of infinity in Ω for any $\lambda > 1$. Moreover, in the symmetric case, if $W > 0$, then the spectrum and the essential spectrum of $W^{-1}P$ are equal to $[1, \infty)$.

Our method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators on a general domain D or on a noncompact manifold. The constructed weight W is given by an explicit simple formula involving two positive solutions of the equation $Pu = 0$.

References

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