

# On a problem of Huang concerning best constants in Sobolev embeddings

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We are concerned with the following Dirichlet problem for an elliptic PDE driven by the  $p$ -Laplacian operator:

$$(P_q) \quad \begin{cases} -\Delta_p u = \lambda |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where  $\Omega \subset \mathbf{R}^N$  ( $N \geq 2$ ) is a bounded smooth domain,  $p > 1$ ,  $0 < q < p^*$  are real numbers and  $\lambda > 0$  is a parameter. Problem  $(P_q)$  is strictly related to the best constants in Sobolev embeddings. We set

$$\inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\|\nabla u\|_p^p}{\|u\|_q^p} = \lambda_q \in ]0, +\infty[.$$

Then, for  $\lambda = \lambda_q$ , problem  $(P_q)$  admits a positive solution  $u_q \in W_0^{1,p}(\Omega)$  s.t.  $\|\nabla u_q\|_p = \lambda_q^{\frac{1}{p}}$  and  $\|u_q\|_q = 1$ . Lee [3] has studied the asymptotic properties of  $u_q$  and  $\lambda_q$  as functions of  $q$ . Later, Huang [2] proved that the mapping  $q \mapsto \lambda_q$  is continuous in  $]1, p[$  and upper semi-continuous in  $]p, p^*[$ , and put forward the conjecture that such mapping “has some monotonicity with respect to  $q$ ”. We both answer such question and improve the continuity results of [2], as we prove what follows ( $|\Omega|$  denotes the Lebesgue  $N$ -dimensional measure of  $\Omega$ ):

**Theorem.** *The mapping  $q \mapsto \lambda_q |\Omega|^{\frac{p}{q}}$  is Lipschitz continuous on compact subsets of  $]0, p^*[$  and decreasing in  $]0, p^*[$ .*

The main novelty with respect to other results in such area (as the interesting paper of Ercole [1]) is that we deal also with the case  $0 < q < 1$ , in which problem  $(P_q)$  is singular. We also prove that, if  $p < N$ , then  $q \mapsto \lambda_q |\Omega|^{\frac{p}{q}}$  is absolutely continuous on  $]0, p^*[$ .

Joint work with G. Anello and F. Faraci.

## References

- [1] *G. Ercole*: Absolute continuity of the best Sobolev constant. *J. Math. Anal. Appl.* 404 (2013) 420–428.
- [2] *Y. X. Huang*: A note on the asymptotic behavior of positive solutions for some elliptic equation. *Nonlinear Anal.* it 29 (1997) 533–537.
- [3] *J. R. Lee*: Asymptotic behavior of positive solutions of the equation  $-\Delta u = \lambda u^p$  as  $p \rightarrow 1$ . *Commun. Partial Differ. Equations* 20 (1995) 633–646.