Sharp constants in Sobolev trace inequalities in BV

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Assume that Ω is a domain, namely a bounded connected open set in $\mathbb{R}^n$, $n \geq 2$, with a sufficiently regular boundary $\partial\Omega$. Then a linear operator if defined on the space $BV(\Omega)$ of functions of bounded variation in Ω, which associates with any function $u \in BV(\Omega)$ its (suitably defined) boundary trace $\tilde{u} \in L^1(\partial\Omega)$. Moreover, a Poincaré type inequality holds, which asserts that there exists a constant $C_1(\Omega)$, depending on Ω, such that

$$\inf_{c \in \mathbb{R}} \|\tilde{u} - c\|_{L^1(\partial\Omega)} \leq C_1(\Omega)\|Du\|(\Omega) \tag{1}$$

for every $u \in BV(\Omega)$, where $\|Du\|(\Omega)$ stands for the total variation over Ω of the distributional gradient Du of u (see e.g. [3]). Here, $C_1(\Omega)$ denotes the optimal (i.e. smallest possible) constant in (1).

Another customary Poincaré trace inequality tells us that

$$\|\tilde{u} - \tilde{u}_{\partial\Omega}\|_{L^1(\partial\Omega)} \leq C_2(\Omega)\|Du\|(\Omega) \tag{2}$$

for every $u \in BV(\Omega)$, where $\tilde{u}_{\partial\Omega}$ stands for the mean value of $\tilde{u}$ over $\partial\Omega$, and $C_2(\Omega)$ denotes the optimal constant in (2).

We are concerned with the problems of minimizing the trace constants $C_1(\Omega)$ and $C_2(\Omega)$, as Ω ranges in the class of all (sufficiently regular) domains Ω in $\mathbb{R}^n$. We prove that both constants attain their minimum value when Ω is a ball [2], a set for which the constants $C_1(\Omega)$ and $C_2(\Omega)$ can be explicitly computed, and extremal functions u in inequalities (1) and (2) can be determined [1]. Moreover, balls are shown to be the only minimizers for $C_1(\Omega)$ for every $n \geq 2$. They are the only minimizers also for $C_2(\Omega)$ when $n \geq 3$, but, interestingly enough, minimizers Ω for $C_2(\Omega)$ can be exhibited which are not (equivalent to) disks if $n = 2$.

Our approach relies upon characterizations of $C_1(\Omega)$ and $C_2(\Omega)$ as genuinely geometric quantities associated with Ω, which are related to certain isoperimetric inequalities relative to Ω.

References

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[3] *V.G. Maz'ya*: Sobolev spaces with applications to elliptic partial differential equations. Springer, Heidelberg, 2011.