Nonlinear Schrödinger equation on graphs

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I will discuss several results related to the nonlinear Schrödinger equation (NLS) on graphs.

I will focus attention on a very simple type of graph with only one vertex and N unlimited edges. This is sometimes referred to as *star-graph*, and one can think of it as the metric space obtained by N half-lines (the edges) with identification of the origins (the vertex). Then posing the NLS on a star-graph is equivalent to consider N nonlinear equations on the half-line coupled by suitable gluing conditions in the vertex.

I will discuss only the gluing condition of δ type with coupling constant α. This can be viewed as the limit of a suitably scaled attractive ($\alpha < 0$) or repulsive ($\alpha > 0$) potential well localized around the vertex. It is worth noticing that for $N = 2$ this model reduces to the one-dimensional NLS with a Dirac-delta potential with coupling constant α. Then the case $\alpha = 0$, sometimes called Kirchhoff condition, plays a special role as natural generalization of the NLS on the real-line to a star-graph.

I will consider only NLS with focusing nonlinearity of power type. On the real-line this is well known to admit stationary solutions which, by virtue of rotational/translational invariance of the equation, generate traveling self-sustained waves (solitons). On the star-graph stationary solutions still exist. However, since the presence of the vertex breaks the translational invariance, in general it is not possible to construct traveling waves.

For any value of α, I will show explicit formulae for the stationary states of the NLS on a star-graph. For $\alpha = 0$ and N even I will exhibit traveling solutions. Then I will discuss our main result. It concerns existence and stability properties of the ground state defined as the minimizer of the constrained action (or energy) functional. We proved that, for subcritical nonlinearity, there is a threshold $\alpha^* < 0$ such that for $\alpha < \alpha^*$ the ground state exists and is orbitally stable. It is a remarkable fact that the threshold value α^* is not observed (i.e. $\alpha^* = 0$) for the corresponding minimization problem on the real-line. I will show how to relate this effect with the presence of $N > 2$ edges.

This work is the result of a collaboration with R. Adami, D. Finco and D. Noja.

References

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