On the scale-invariant critical Hardy's inequality and related variational problems

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The standard Hardy inequality states that

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbf{R}} \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \le \int_{\mathbf{R}} |\nabla u|^p\,\mathrm{d}x \tag{1}$$

holds for every $u \in W^{1,p}(\mathbf{R})$, where $N \geq 2$ and $1 \leq p < N$. This type of inequality arises in vast area of mathematical sciences, e.g, the verification of the stability of a hydrogen atom. In this talk, we focus on the critical case $p = N$ in the unit ball $B_1 := \{x \in \mathbf{R} \colon |x| < 1\}$. In this case, it is known that the inequality

$$\left(\frac{N-1}{N}\right)^N \int_{B_1} \frac{|u(x)|^N}{|x|^N\left(1+\log\frac{1}{|x|}\right)^N} \le \int_{B_1} |\nabla u(x)|^N\,\mathrm{d}x, \quad u \in W_0^{1,N}(B_1) \tag{2}$$

holds. Indeed, the critical inequality of the type (2) in a bounded domain which contains the origin is known (see [1]).

In spite of the natural similarity between the critical inequality (2) and the original inequality (1), there exists a crucial difference between them. Namely, there seem to exist no natural scale invariance properties for the critical inequality (2) and its generalization with remainder terms, while the original inequality (1) is invariant under the scaling $u_\lambda(x) = \lambda^{\frac{N-p}{p}} u(\lambda x)$. In this talk, we introduce a generalization of the critical inequality (2) which is invariant under a certain scaling and discuss related variational problems. This is a joint work with Prof. Ioku in Ehime university.

References

[1] *D. E. Edmunds, H. Triebel*: Sharp Sobolev embeddings and related Hardy inequalities: The critical case. Math. Nachr. *207* (1999), 79–92.

[2] *N. Ioku, M. Ishiwata*: Scale invariant form of critical Hardy's inequality. Preprint.