Finite-time blow-up in parabolic Keller-Segel systems

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We study the blow-up phenomenon for parabolic chemotaxis models, in particular the Neumann initial-boundary value problem for the fully parabolic Keller-Segel system

$$(\star) \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

in a ball $\Omega \subset \mathbb{R}^n$ with $n \geq 2$.

The main question pursued in the presentation is in how far blow-up in finite time can be viewed as a generic phenomenon in $(\star)$. For this purpose, we present essentially explicit conditions on the initial data which are sufficient for such a finite-time singularity formation. A particular focus is on the method by which these criteria are derived: In both cases $n \geq 3$ and $n = 2$, it is based on an appropriate estimate from below for the natural energy functional associated with $(\star)$,

$$\mathcal{F}(u,v) := \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u,$$

in terms of the corresponding dissipation rate

$$\mathcal{D}(u,v) := \int_\Omega |\Delta - v + u|^2 + \int_\Omega \Big| \frac{\nabla u}{\sqrt{u}} - \sqrt{u} \nabla v \Big|^2.$$

This is achieved by deriving suitable a priori estimates for radial solutions of the elliptic-hyperbolic system

$$\begin{cases} -\Delta v + v = u + f, & x \in \Omega, \\ \frac{\nabla u}{\sqrt{u}} - \sqrt{u} \nabla v = g, & x \in \Omega, \end{cases}$$

in dependence of the inhomogeneities f and g.

The results for $n = 2$ were derived in collaboration with Noriko Mizoguchi.