A closer look at conservative and dissipative solutions for the Camassa-Holm equation

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There is a well-known and well-studied dichotomy between two distinct and distinguished classes of solutions of the Camassa-Holm (CH) equation. The two classes appear exactly at wave breaking where the spatial derivative of the solution becomes unbounded while its $H^1(\mathbb{R})$ norm remains finite. We here introduce a novel solution concept gauged by a continuous parameter α in such a way that $\alpha = 0$ corresponds to conservative solutions and $\alpha = 1$ gives the dissipative solutions. This allows for a detailed study of the difference between the two classes of solutions.

More precisely, we consider the Cauchy problem given by (κ is a given real constant)

$$u_t - u_{txx} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0,$$

with initial data $u|_{t=0} = u_0 \in H^1(\mathbb{R})$.

The equation enjoys blow-up in finite time, even for smooth initial data. Continuation of the solution past blow-up is intricate. It has turned out to be two distinct ways to continue the solution past blow-up, denoted conservative solutions, associated with preservation of the $H^1(\mathbb{R})$ norm, and dissipative solutions, characterized by a sudden drop in the $H^1(\mathbb{R})$ norm at blow-up. The upshot of the analysis is that the solution u has to be augmented by an additional variable in the form of a measure μ that describes the energy. To this end we consider a nonnegative, finite Radon measure μ such that the absolutely continuous part μ_{ac} satisfies $\mu_{\mathrm{ac}} = u_x^2 \, \mathrm{d}x$. At wave breaking the energy, as measured by μ, transfers to the singular part of μ, while the measure remains absolutely continuous between the times of wave breaking.

We analyze in detail the relation between the conservative and dissipative solutions. This is joint work with Katrin Grunert (NTNU) and Xavier Raynaud (Sintef).