Optimal initial values and regularity conditions of Besov space type for weak solutions to the Navier-Stokes system

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In this talk we present recent results jointly obtained with Hermann Sohr (Paderborn) and W. Varnhorn (Kassel).

The first question concerns the *optimal condition* on initial values $u_0 \in L^2_\sigma(\Omega)$ to get a locally regular solution to the instationary Navier-Stokes system in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, i.e., an assumption on u_0 such that the corresponding weak solution u satisfies Serrin's regularity condition $u \in L^s(0,T;L^q(\Omega))$ where $\frac{2}{s} + \frac{3}{q} = 1$, $s > 2$, $q > 3$. It is shown that the condition

$$\int_0^\infty \|e^{-\tau A} u_0\|^s_{L^q(\Omega)} \, \mathrm{d}\tau < \infty \tag{1}$$

is *necessary and sufficient* for this local in time regularity result; here A denotes the Stokes operator on $L^2_\sigma(\Omega)$. Condition (1) is weaker than the more classical assumptions $u_0 \in H^1_0(\Omega)$, $u_0 \in \mathcal{D}(A^{1/4})$ or $u_0 \in L^3(\Omega)$ and ensures that the solution $e^{-\tau A}u_0$ of the linear Stokes problem lies in Serrin's class $L^s(0,\infty;L^q(\Omega))$. It can be rewritten in the form

$$u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega)$$

where $\mathbb{B}^{-2/s}_{q,s}(\Omega)$ denotes a solenoidal subspace of the usual Besov space $B^{-2/s}_{q,s}(\Omega)$.

Actually, to obtain local in time regularity, it suffices to consider the integral in (1) only on some finite time interval $(0,\delta)$, $\delta > 0$, leading to a Besov space $\mathbb{B}^{-2/s}_{q,s;(\delta)}(\Omega)$, with equivalent norm. Using this space we find new regularity (and uniqueness) criteria for weak Leray-Hopf solutions. E.g., if $u \in L^\infty(0,T;\mathbb{B}^{-2/s}_{q,s;(\delta)}(\Omega))$, we get the implication

$$\lim_{\delta \to 0} \|u\|_{L^\infty(0,T;\mathbb{B}^{-2/s}_{q,s;(\delta)}(\Omega))} = 0 \quad \Rightarrow \quad u \text{ is regular.}$$

Several of these results can also be obtained for exterior domains and general unbounded domains where the space $L^q(\Omega)$, $2 < q < \infty$, must be replaced by $L^q(\Omega) \cap L^2(\Omega)$.

References

[1] *R. Farwig, H. Sohr, W. Varnhorn*: Besov space regularity conditions for weak solutions of the Navier-Stokes equations. Preprint.

[2] *R. Farwig, H. Sohr, W. Varnhorn*: On optimal initial value conditions for local strong solutions of the Navier-Stokes equations. Ann. Univ. Ferrara *55* (2009), 89–110.