An attempt to create fast numerical schemes with the discrete variational derivative method

Daisuke Furihata
Cybermedia Center, Osaka University, Japan
furihata@cmc.osaka-u.ac.jp

The discrete variational derivative method is a structure-preserving numerical method for partial differential equations. We indicate a typical example of the target equations:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^m \left(\frac{\delta G}{\delta u}\right), \quad m \geq 0,$$

for an unknown function $u = u(x,t)$ where x and t indicate space and time, respectively. The $\delta G/\delta u$ is a variational derivative of G for u. These systems have a remarkable quantity $J[u] \stackrel{\text{def}}{=} \int G \,\mathrm{d}x$. It is conservative when m is odd and dissipative when m is even. The obtained numerical schemes mimic this variational structure of the original equations, and they inherit the property of the quantity J in discrete context. We have applied this method to some nonlinear partial differential equations such as the Cahn-Hilliard equation, the nonlinear Schrödinger equation and the Korteweg de Vries equation. The derived schemes are stable, and the obtained numerical solutions are "good" from various viewpoints.

The designed schemes are nonlinear when the original equations are nonlinear, and this nonlinearity of schemes may be a severe computational difficulty. To overcome this difficulty, we have developed an extension of the discrete variational derivative method, a.k.a. the linearization technique. Based on the linearization technique we are able to design some linear numerical schemes for nonlinear partial differential equations. Those linear schemes also conserve some relaxed conservation properties or dissipation ones, but they tend to be unstable. Furthermore, the linearization technique is applicable to only lower order polynomial equations.

In such a situation, we extended the linearization technique to be applicable to nonpolynomial problems. This means that we are able to obtain fast numerical schemes with the discrete variational derivative method for every equation. However, we do not improve the unstable tendency of the obtained schemes. We, therefore, attempt a breakthrough based on the classical predictor-corrector iteration method to avoid this tendency.

References

[1] *D. Furihata, T. Matsuo*: Discrete Variational Derivative Method: A Structure-preserving Numerical Method for Partial Differential Equations. CRC Press, Florida, 2010.

[2] *D. Furihata, T. Matsuo*: A stable, convergent, conservative and linear finite difference scheme for the Cahn-Hilliard equation. Japan J. Ind. Appl. Math. *20* (2003), 65–85.