

Necessary and sufficient conditions for the existence of Helmholtz decompositions in general domains

Werner Varnhorn

Institute of Mathematics, Kassel University, Germany

varnhorn@mathematik.uni-kassel.de

Consider a general domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, and let $1 < q < \infty$. Our first result is based on the estimate for the gradient $\nabla p \in G^q(\Omega)$ in the form $\|\nabla p\|_q \leq C \sup |\langle \nabla p, \nabla v \rangle_\Omega| / \|\nabla v\|_{q'}$, $\nabla v \in G^{q'}(\Omega)$, $q' = \frac{q}{q-1}$, with some constant $C = C(\Omega, q) > 0$. This estimate was introduced in [1] for smooth bounded and exterior domains. We show for general domains that the validity of this gradient estimate in $G^q(\Omega)$ and in $G^{q'}(\Omega)$ is necessary and sufficient for the validity of the Helmholtz decomposition in $L^q(\Omega)$ and in $L^{q'}(\Omega)$. A new aspect concerns the estimate for divergence free functions $f_0 \in L^q_\sigma(\Omega)$ in the form $\|f_0\|_q \leq C \sup |\langle f_0, w \rangle_\Omega| / \|w\|_{q'}$, $w \in L^{q'}_\sigma(\Omega)$, for the second part of the Helmholtz decomposition. We show again for general domains that the validity of this estimate in $L^q_\sigma(\Omega)$ and in $L^{q'}_\sigma(\Omega)$ is necessary and sufficient for the validity of the Helmholtz decomposition in $L^q(\Omega)$ and in $L^{q'}(\Omega)$. This is a joint work with Christian G. Simader and Hermann Sohr.

References

- [1] *C. G. Simader, H. Sohr*: A new approach to the Helmholtz decomposition and the Neumann problem in L^q -spaces for bounded and exterior domains. Ser. Adv. Math. Appl. Sci. 11, 1–35, World Scientific, Singapore, 1992.