Periodic solutions of a fluid particle induced by a prescribed vortex path in a circular domain

Pedro J. Torres

Departamento de Matemática Aplicada, Universidad de Granada, Spain

ptorres@ugr.es, http://www.ugr.es/∼ptorres/

This is a joint work with Alberto Boscaggin (S.I.S.S.A.). We consider the motion of a two-dimensional ideal fluid in a circular domain of radius $R > 0$ subjected to the action of a moving point vortex whose position, denoted as $z(t)$, is a prescribed T-periodic function of time. This model plays an important role in Fluid Mechanics as an idealized model of the stirring of a fluid inside a cylindrical tank by an agitator. A fundamental reference for this problem is the seminal paper [1], where the concept of *chaotic advection* was coined. Following the classical Lagrangian representation, the mathematical model under consideration is the planar system

$$\dot{\bar{\zeta}} = \frac{\Gamma}{2\pi i}\Big(\frac{|z(t)|^2 - R^2}{(\zeta - z(t))(\zeta\bar{z}(t) - R^2)}\Big), \tag{1}$$

where the complex variable ζ represents the particle transport induced by the so-called *stirring protocol* $z(t)$. System (1) is a T-periodically forced planar system with hamiltonian structure, where the stream function

$$\Psi(t,\zeta) = \frac{\Gamma}{2\pi}\ln\Big|\frac{\zeta - z(t)}{\bar{z}(t)\zeta - R^2}\Big|$$

plays the role of the hamiltonian. Observe the moving singularity in $z(t)$.

The main contribution of Aref in [1] was to show that the flow may experience regular or chaotic regimes depending on the particular stirring protocol. For instance, system (1) is integrable if $z(t)$ is constant or $z(t) = z_0 \exp(i\Omega t)$ but it is chaotic if $z(t)$ is piecewise constant (blinking protocol in the related literature). More recently, other strategies of stirring have been studied, for instance the figure-eight or the epitrochoidal protocol [2], but only from a numerical point of view. Our contribution in this paper is to prove that both regular and chaotic regimes share a common dynamical feature, namely the existence of an infinite number of periodic solutions labeled by the number of revolutions around the vortex in the course of a period. The proof is based on Poincaré-Birkhoff Theorem.

References

[1] *H. Aref*, Stirring by chaotic advection. J. Fluid Mech. *143* (1984), 1–21.

[2] *S. Wiggins, J. M. Ottino*, Foundations of chaotic mixing. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. *362* (2004), 937–970.