Heat flow, optimal transport and curvature

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It is well known that the heat flow in $\mathbf{R}^n$ can be considered as the gradient flow of the Dirichlet energy in $L^2(\mathbf{R}^n)$. The theory of Dirichlet forms in a Lebesgue space $L^2(X, \mathfrak{m})$ greatly extends this variational point of view to a general setting. Exploiting locality and diffusion, a powerful Γ-calculus has been developed, with a strong connection with the geometric properties of the ambient space X, in particular with lower Ricci curvature bounds, that can be expressed through the Bakry-Émery [5] *Carré du champ itéré* Γ_2 and the celebrated curvature-dimension condition $\mathrm{CD}(K, N)$.

After the pioneering paper by Jordan-Kinderlehrer-Otto [6] a new characterization of the heat flow has been introduced: it involves the logarithmic entropy functional and the optimal transportation distance between probability measures [9], [1] and it has been considerably extended to general metric-measure spaces $(X, \mathsf{d}, \mathfrak{m})$. Entropy and transport are also the basic tools of the synthetic notion of Ricci curvature-dimension bounds of Lott-Sturm-Villani [7], [8], that provides powerful geometric-functional inequalities and is stable w.r.t. measured Gromov-Hausdorff convergence.

The talk will discuss the recent progresses [2], [3], [4] (in collaboration with L. Ambrosio and N. Gigli) towards the unification of the two points of view.

References

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