Differentiability properties of p-trigonometric functions

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The p-trigonometric functions are generalizations of trigonometric functions. They appear in the context of nonlinear differential equations and also in analytical geometry of the p-circle in the plane. The most important p-trigonometric function is $\sin_p(x)$. For given $p > 1$, this function is defined for any $x \in \mathbb{R}$ as the unique solution of the initial value problem

$$(|u'(x)|^{p-2}u'(x))' = (p-1)|u(x)|^{p-2}u(x),\ u(0) = 0,\ u'(0) = 1.$$

We proved that the n-th derivative of $\sin_p(x)$ can be expressed in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x),$$

on $(0, \frac{\pi_p}{2})$ where $\pi_p = \int_0^1 (1 - s^p)^{-1/p}\, \mathrm{ds}$, and $\cos_p(x) = \sin_p'(x)$. Using this formula, we proved the order of differentiability of the function $\sin_p(x)$. The most surprising (least expected) result is that $\sin_p(x) \in C^\infty(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ if p is an even integer. This result was essentially used in the proof of a theorem, which says that the Maclaurin series of $\sin_p(x)$ converges on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$ if p is an even integer. This completes previous results that were known e.g. by Lindqvist and Peetre where this convergence was conjectured. This is a joint work with Petr Girg.

References

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